

# EXPLICIT MATRICES FOR HECKE OPERATORS ON SIEGEL MODULAR FORMS

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ABSTRACT. We present an explicit set of matrices giving the action of the Hecke operators  $T(p), T_j(p^2)$  on Siegel modular forms.

## INTRODUCTION

It is well-known that the space of elliptic modular forms of weight  $k$  has a basis of simultaneous eigenforms for the Hecke operators, and the Fourier coefficients of an eigenform (and hence the eigenform) are completely determined by its eigenvalues and first Fourier coefficient. In the theory of Siegel modular forms, the role of the Hecke operators is not yet completely understood, thus there are many avenues open for conjecture and exploration, including computational exploration. The purpose of this note is to present an explicit set of matrices giving the action of the Hecke operators on Siegel modular forms, with the goal of facilitating computational exploration. (This construction also yields an explicit set of matrices giving the action of Hecke operators on Jacobi modular forms; we remark on this further at the end of this note.)

## DEFINITIONS AND RESULTS

For  $F$  a Siegel modular form of degree  $n$  and  $p$  a prime, we define the Hecke operator  $T(p)$  by

$$F|T(p) = p^{n(k-n-1)/2} \sum_M F| \begin{pmatrix} \frac{1}{p}I_n & \\ & I_n \end{pmatrix} M$$

where  $M$  runs over a complete set of coset representatives for  $(\Gamma' \cap \Gamma) \backslash \Gamma$ , with  $\Gamma = Sp_n(\mathbb{Z})$  and

$$\Gamma' = \begin{pmatrix} pI_n & \\ & I_n \end{pmatrix} \Gamma \begin{pmatrix} \frac{1}{p}I_n & \\ & I_n \end{pmatrix}.$$

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1991 *Mathematics Subject Classification.* 11F41.

*Key words and phrases.* Siegel modular forms; Hecke operators; Jacobi modular forms.

Similarly, for  $1 \leq j \leq n$ , we define  $T_j(p^2)$  by

$$F|T_j(p^2) = \sum_M F| \begin{pmatrix} \frac{1}{p}I_j & & & \\ & I_{n-j} & & \\ & & pI_j & \\ & & & I_{n-j} \end{pmatrix} M$$

where  $M$  runs over a complete set of coset representatives for  $(\Gamma'_j \cap \Gamma) \backslash \Gamma$ ; here

$$\Gamma'_j = \begin{pmatrix} pI_j & & & \\ & I_{n-j} & & \\ & & \frac{1}{p}I_j & \\ & & & I_{n-j} \end{pmatrix} \Gamma \begin{pmatrix} \frac{1}{p}I_j & & & \\ & I_{n-j} & & \\ & & pI_j & \\ & & & I_{n-j} \end{pmatrix}.$$

In [3], we determine the action of Hecke operators on Fourier coefficients of a Siegel modular form by first describing a complete set of coset representatives for the Hecke operators. We index the cosets using lattices; the coset representatives are then explicitly described except for a choice of  $G \in GL_n(\mathbb{Z})$  associated to each lattice. There are infinitely many possible choices for each  $G$ ; in this note we make an explicit choice for each  $G$ .

We first construct matrices for  $T_j(p^2)$  (and for the averaged operators  $\tilde{T}_j(p^2)$  introduced in [3]); then we do the same for  $T(p)$ .

For  $T_j(p^2)$ , we construct these  $G$  as follows. For (nonnegative) integers  $r_0, r_2$  with  $r_0 + r_2 \leq j$  and  $r_1 = j - r_0 - r_2$ , we call  $\mathcal{P}$  a partition of type  $(r_0, r_2)$  for  $(n, j)$  if  $\mathcal{P}$  is an ordered partition

$$(\{d_1, \dots, d_{r_0}\}, \{b_1, \dots, b_{r_1}\}, \{a_1, \dots, a_{r_2}\}, \{c_1, \dots, c_{n-j}\})$$

of  $\{1, 2, \dots, n\}$ . (Note that if some  $r_i = 0$  or  $n - j = 0$ , a set in the partition could be empty.) Given a partition  $\mathcal{P}$  of type  $(r_0, r_2)$ , we let  $\mathcal{G}_{\mathcal{P}} \subseteq GL_n(\mathbb{Z})$  consist of all matrices  $G = (G_0, G_1, G_2, G_3)$  constructed as follows.  $G_0$  is the  $n \times r_0$  matrix with  $\ell, t$ -entry 1 if  $\ell = d_t$ , and 0 otherwise.  $G_1$  is an  $n \times r_1$  matrix with  $\ell, t$ -entry  $\beta_{\ell t}$  where  $\beta_{\ell t} = 1$  if  $\ell = b_t$ ,  $\beta_{\ell t} = 0$  if  $\ell < b_t$  or  $\ell = a_i$  (some  $i$ ) or  $\ell = b_i$  (some  $i \neq t$ ), and otherwise  $\beta_{\ell t} \in \{0, 1, \dots, p-1\}$ .  $G'_2$  is an  $n \times r_2$  matrix with  $\ell, t$ -entry  $\alpha_{\ell t}$  where  $\alpha_{\ell t} = 1$  if  $\ell = a_t$ ,  $\alpha_{\ell t} = 0$  if  $\ell < a_t$  or  $\ell = a_i$  (some  $i \neq t$ ), and otherwise  $\alpha_{\ell t} \in \{0, 1, \dots, p-1\}$ .  $G''_2$  is an  $n \times r_2$  matrix with  $\ell, t$ -entry  $\delta_{\ell t}$  where  $\delta_{\ell t} = 0$  if  $\ell \neq d_i$  (any  $i$ ), and otherwise  $\delta_{\ell t} \in \{0, 1, \dots, p-1\}$ .  $G_2 = G'_2 + pG''_2$ .  $G_3$  is an  $n \times (n - j)$  matrix with  $\ell, t$ -entry  $\gamma_{\ell t}$  where  $\gamma_{\ell t} = 1$  if  $\ell = c_t$ ,  $\gamma_{\ell t} = 0$  if  $\ell < c_t$  or  $\ell = a_i$  or  $b_i$  (some  $i$ ) or  $\ell = c_i$  (some  $i \neq t$ ), and otherwise  $\gamma_{\ell t} \in \{0, 1, \dots, p-1\}$ .

Note that  $(G_0, G_1, G'_2, G_3)$  is a (column) permutation of an integral lower triangular matrix with 1's on the diagonal, and thus is an element of  $GL_n(\mathbb{Z})$ . Also, it is easy to see that there is an elementary matrix  $E$  so that

$$(G_0, G_1, G'_2, G_3)E = (G_0, G_1, G'_2 + pG''_2, G_3) = G,$$

and so  $G \in GL_n(\mathbb{Z})$ . (After proving Theorem 1, we describe  $G^{-1}$  as a product of four explicit matrices.)

We let  $\mathcal{G}_{r_0, r_2} = \cup_{\mathcal{P}} \mathcal{G}_{\mathcal{P}}$  where  $\mathcal{P}$  varies over all partitions of type  $(r_0, r_2)$ . We set

$$D_{r_0, r_2} = \begin{pmatrix} I_{r_0} & & & \\ & pI_{r_1} & & \\ & & p^2I_{r_2} & \\ & & & I_{n-j} \end{pmatrix}.$$

Also, we let  $\mathcal{Y}_{r_0, r_2}$  be the set of all (integral) matrices of the form

$$\begin{pmatrix} Y_0 & Y_2 & 0 & Y_3 \\ p {}^tY_2 & Y_1 & & \\ 0 & & & \\ {}^tY_3 & & & \end{pmatrix}$$

where  $Y_0$  is symmetric,  $r_0 \times r_0$ , with entries varying modulo  $p^2$ ,  $Y_1$  is symmetric,  $r_1 \times r_1$ , with entries varying modulo  $p$ ,  $Y_2$  is  $r_0 \times r_1$  with entries varying modulo  $p$ , and  $Y_3$  is  $r_0 \times (n - j)$  with entries varying modulo  $p$ . We let  $\mathcal{Y}'_{r_0, r_2}$  be those matrices in  $\mathcal{Y}_{r_0, r_2}$  that satisfy the additional condition  $p \nmid \det Y_1$  (which is trivially satisfied if  $r_1 = 0$ ).

**Theorem 1.** *Given a degree  $n$  Siegel modular form  $F$ ,  $1 \leq j \leq n$ ,*

$$F|T_j(p^2) = \sum_M F| \begin{pmatrix} \frac{1}{p}I_j & & & \\ & I_{n-j} & & \\ & & pI_j & \\ & & & I_{n-j} \end{pmatrix} M$$

where

$$M = \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

varies so that for some  $r_0, r_2$  with  $r_0 + r_2 \leq j$ ,  $D = D_{r_0, r_2}$ ,  $Y \in \mathcal{Y}'_{r_0, r_2}$ , and  $G \in \mathcal{G}_{r_0, r_2}$ . Also, with

$$\tilde{T}_j(p^2) = p^{j(k-n-1)} \sum_{0 \leq t \leq j} \begin{bmatrix} n-t \\ j-t \end{bmatrix}_p T_t(p^2)$$

where  $\begin{bmatrix} m \\ r \end{bmatrix}_p = \prod_{i=0}^{r-1} \frac{p^{m-i}-1}{p^{r-i}-1}$ ,

$$F|\tilde{T}_j(p^2) = p^{j(k-n-1)} \sum_M F| \begin{pmatrix} \frac{1}{p}I_j & & & \\ & I_{n-j} & & \\ & & pI_j & \\ & & & I_{n-j} \end{pmatrix} M$$

where

$$M = \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

varies so that for some  $r_0, r_2$  with  $r_0 + r_2 \leq j$ ,  $D = D_{r_0, r_2}$ ,  $Y \in \mathcal{Y}_{r_0, r_2}$ , and  $G \in \mathcal{G}_{r_0, r_2}$ .

*Proof.* As mentioned above, in Proposition 2.1 of [3] we found a complete set of coset representatives indexed by lattices  $(\Omega, \Lambda_1)$  where, for  $\Lambda$  a fixed lattice of rank  $n$ ,  $\Omega$  varies subject to  $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$ , and  $\overline{\Lambda}_1$  varies over all codimension  $n - j$  subspaces of  $\Lambda \cap \Omega / p(\Lambda + \Omega)$ . With  $\{\Lambda : \Omega\}$  denoting the invariant factors of  $\Omega$  in  $\Lambda$  and  $r_0$  the multiplicity of the invariant factor  $p$ ,  $r_2$  the multiplicity of the invariant factor  $\frac{1}{p}$ , the action of the coset representatives corresponding to  $(\Omega, \Lambda_1)$  is given by the matrices

$$\begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^tG \end{pmatrix}$$

where  $D = D_{r_0, r_2}$ ,  $Y$  varies over  $\mathcal{Y}'_{r_0, r_2}$ , and  $G = G(\Omega, \Lambda_1)$  is any change of basis matrix so that, relative to a fixed basis  $(x_1, \dots, x_n)$  for  $\Lambda$ ,

$$\Omega = \Lambda G D^{-1} \begin{pmatrix} pI_j & \\ & I_{n-j} \end{pmatrix}, \quad \Lambda_1 = \Lambda G \begin{pmatrix} 0_{r_0} & & \\ & I_{r_1} & \\ & & 0 \end{pmatrix}$$

where  $r_1 = j - r_0 - r_2$ . (Thus only those  $\Omega$  occur where  $r_0 + r_2 \leq j$ .) So we need to show that such a pair  $(\Omega, \Lambda_1)$  corresponds to a unique  $G \in \mathcal{G}_{r_0, r_2}$ . We do this by constructing all  $(\Omega, \Lambda_1)$  for each pair of parameters  $(r_0, r_2)$ , simultaneously building a partition  $\mathcal{P}$  and making choices of  $\alpha_{\ell t}, \beta_{\ell t}, \gamma_{\ell t}, \delta_{\ell t}$  so that with  $G$  constructed according to our recipe above, we can take  $G$  for  $G(\Omega, \Lambda_1)$ .

Notice that when  $p\Lambda \subseteq \Omega \subseteq \frac{1}{p}\Lambda$ , the Invariant Factor Theorem (81:11 of [5]) tells us we have compatible decompositions:

$$\begin{aligned} \Lambda &= \Lambda_0 \oplus \Lambda'_1 \oplus \Lambda_2, \\ \Omega &= p\Lambda_0 \oplus \Lambda'_1 \oplus \frac{1}{p}\Lambda_2. \end{aligned}$$

On the other hand, given  $\Lambda$ , such an  $\Omega$  is determined by  $\Omega' = \Lambda_2 + p\Lambda$  and  $(p\Lambda'_1 \oplus \Lambda_2) + p\Omega'$ . Also, in  $\Lambda \cap \Omega / p(\Lambda + \Omega)$ ,  $\overline{\Lambda}_2 = 0$ , so  $\Lambda_1$  can be chosen so that in  $\Omega' / p\Omega'$ ,  $\overline{\Lambda}_1 \subseteq \overline{p\Lambda'_1} \subseteq \overline{p\Lambda}$ .

So to begin our construction of  $\Omega, \Lambda_1$  and  $G = G(\Omega, \Lambda_1)$ , in  $\Lambda / p\Lambda$  we choose a dimension  $r_2$  subspace  $\overline{C}'$ ; let  $(\overline{v}'_1, \dots, \overline{v}'_{r_2})$  be a basis for  $\overline{C}'$ . Each  $\overline{v}'_t$  is a linear combination over  $\mathbb{Z}/p\mathbb{Z}$  of the  $\overline{x}_i$ ; by adjusting the  $\overline{v}'_t$  we can assume

$$\overline{v}'_t = \overline{x}_{a_t} + \sum_{\ell > a_t} \overline{\alpha}_{\ell t} \overline{x}_\ell$$

where  $a_1, \dots, a_{r_2}$  are distinct and  $\overline{\alpha}_{\ell t} = 0$  if  $\ell = a_i$  (some  $i \neq t$ ). Let  $\alpha_{\ell t} \in \{0, 1, \dots, p-1\}$  be a preimage of  $\overline{\alpha}_{\ell t}$ .

Now let  $\Omega'$  be the preimage in  $\Lambda$  of  $\overline{C}'$ . In  $\Omega' / p\Omega'$  we will construct a dimension  $n - r_0$  subspace  $\overline{C}$  so that  $\dim(\overline{C} \cap \overline{p\Lambda}) = n - r_0 - r_2$ , distinguishing a dimension  $r_1$

subspace  $\overline{p\Lambda_1}$  of  $\overline{C} \cap \overline{p\Lambda}$ . We begin by choosing  $\overline{p\Lambda_1}$  to be a dimension  $r_1$  subspace of  $\overline{p\Lambda}$ ; let  $\overline{pu_1}, \dots, \overline{pu_{r_1}}$  be a basis for  $\overline{p\Lambda_1}$ . Since  $\overline{px_{a_i}} = 0$  in  $\Omega'/p\Omega'$ , we can adjust the  $\overline{pu_t}$  so that

$$\overline{pu_t} = \overline{px_{b_t}} + \sum_{\ell > b_t} \overline{\beta_{\ell t} px_{\ell}}$$

where  $b_1, \dots, b_{r_1}$  are distinct,  $b_t \neq a_i$  (any  $i$ ), and  $\overline{\beta_{\ell t}} = 0$  if  $\ell = a_i$  (some  $i$ ) or  $\ell = b_i$  (some  $i \neq t$ ). Let  $\beta_{\ell t} \in \{0, 1, \dots, p-1\}$  be a preimage of  $\overline{\beta_{\ell t}}$ .

Now extend  $\overline{p\Lambda_1}$  to a dimension  $n - r_0 - r_2$  subspace  $\overline{p\Lambda'_1}$  of  $\overline{p\Lambda}$  in  $\Omega'/p\Omega'$ . Extend  $(\overline{pu_1}, \dots, \overline{pu_{r_1}})$  to a basis

$$(\overline{pu_1}, \dots, \overline{pu_{r_1}}, \overline{pw_1}, \dots, \overline{pw_{n-j}})$$

for  $\overline{p\Lambda'_1}$  so that

$$\overline{pw_t} = \overline{px_{c_t}} + \sum_{\ell > c_t} \overline{\gamma_{\ell t} px_{\ell}},$$

where  $c_1, \dots, c_{n-j}$  are distinct,  $c_t \neq a_i, b_i$  (any  $i$ ), and  $\overline{\gamma_{\ell t}} = 0$  if  $\ell = a_i$  (some  $i$ ), or  $\ell = b_i$  (some  $i \neq t$ ). Let  $\gamma_{\ell t} \in \{0, 1, \dots, p-1\}$  be a preimage of  $\overline{\gamma_{\ell t}}$ .

Now we extend  $\overline{p\Lambda'_1}$  to a dimension  $n - r_0$  space  $\overline{C}$  so that the dimension of  $\overline{C} \cap \overline{p\Lambda}$  is  $n - r_0 - r_2 = r_1 + n - j$ , and we extend  $(\overline{pu_1}, \dots, \overline{pw_1}, \dots)$  to a basis

$$(\overline{pu_1}, \dots, \overline{pu_{r_1}}, \overline{pw_1}, \dots, \overline{pw_{n-j}}, \overline{pv_1}, \dots, \overline{pv_{r_2}})$$

for  $\overline{C}$ . Taking  $d_1, \dots, d_{r_0}$  so that

$$(\{d_1, \dots, d_{r_0}\}, \{b_1, \dots, b_{r_1}\}, \{a_1, \dots, a_{r_2}\}, \{c_1, \dots, c_{n-j}\})$$

is a partition of  $\{1, \dots, n\}$ , we can take

$$\overline{v_t} = \overline{v'_t} + \sum_{m=1}^{r_0} \overline{\delta_{mt} px_{d_m}}$$

for some  $\overline{\delta_{mt}}$ ; let  $\delta_{mt} \in \{0, 1, \dots, p-1\}$  be a preimage of  $\overline{\delta_{mt}}$ .

Now let  $p\Omega$  be the preimage in  $\Omega'$  of  $\overline{C}$ . So with

$$\begin{aligned} u_t &= x_{b_t} + \sum_{\ell > b_t} \beta_{\ell t} x_{\ell} \quad (1 \leq t \leq r_1), \\ v_t &= x_{a_t} + \sum_{\ell > a_t} \alpha_{\ell t} x_{\ell} + p \sum_m \delta_{mt} x_{d_m} \quad (1 \leq t \leq r_2), \\ w_t &= x_{c_t} + \sum_{\ell > c_t} \gamma_{\ell t} x_{\ell} \quad (1 \leq t \leq n-j), \end{aligned}$$

the vectors

$$(px_{d_1}, \dots, px_{d_{r_0}}, u_1, \dots, u_{r_1}, \frac{1}{p}v_1, \dots, \frac{1}{p}v_{r_2}, w_1, \dots, w_{n-j})$$

form a basis for  $\Omega$ , and  $(\bar{u}_1, \dots, \bar{u}_{r_1})$  is a basis for  $\bar{\Lambda}_1$  in  $\Lambda \cap \Omega/p(\Lambda + \Omega)$ .  $\square$

*Remark.* Given  $G \in \mathcal{G}_{\mathcal{P}}$  as above,  $G^{-1} = E_1 E_2 E_3 E_4$  where the  $E_i$  are  $n \times n$  matrices constructed as follows.  $E_1$  has  $i, i$ -entry 1 ( $1 \leq i \leq n$ ); for  $1 \leq \ell \leq r_0$ ,  $E_1$  has  $\ell, r_0 + t$ -entry  $-\beta_{d_{\ell}t}$  ( $1 \leq t \leq r_1$ ),  $\ell, r_0 + r_1 + t$ -entry  $-\alpha_{d_{\ell}t} - p\delta_{d_{\ell}t}$  ( $1 \leq t \leq r_2$ ),  $\ell, r_0 + r_1 + r_2 + t$ -entry  $-\gamma_{d_{\ell}t}$  ( $1 \leq t \leq n - j$ ), and all other entries 0. So for  $1 \leq t \leq n - j$ , column  $r_0 + r_1 + r_2 + t$  of  $GE_1$  has a 1 in row  $c_t$ , and zeros elsewhere.  $E_2$  has  $i, i$ -entry 1 ( $1 \leq i \leq n$ ); for  $1 \leq \ell \leq n - j$ ,  $E_2$  has  $r_0 + r_1 + r_2 + \ell, r_0 + t$ -entry  $-\beta_{c_{\ell}t}$  ( $1 \leq t \leq r_1$ ),  $r_0 + r_1 + r_2 + \ell, r_0 + r_1 + t$ -entry  $-\alpha_{c_{\ell}t}$  ( $1 \leq t \leq r_2$ ), and zeros elsewhere. Thus for  $1 \leq t \leq r_1$ , column  $r_0 + t$  of  $GE_1 E_2$  has a 1 in row  $b_t$ , and zeros elsewhere.  $E_3$  has  $i, i$ -entry 1 ( $1 \leq i \leq n$ ); for  $1 \leq \ell \leq r_1$ ,  $1 \leq t \leq r_2$ ,  $E_3$  has  $r_0 + \ell, r_0 + r_1 + t$ -entry  $-\alpha_{b_{\ell}t}$  and zeros elsewhere. Thus  $GE_1 E_2 E_3 = {}^t E_4$  is a permutation matrix;  $E_4$  has 1 as its  $\ell, d_{\ell}$ -entry ( $1 \leq \ell \leq r_0$ ), 1 as its  $r_0 + \ell, b_{\ell}$ -entry ( $1 \leq \ell \leq r_1$ ), 1 as its  $r_0 + r_1 + \ell, a_{\ell}$ -entry ( $1 \leq \ell \leq r_2$ ), 1 as its  $r_0 + r_1 + r_2 + \ell, c_{\ell}$ -entry ( $1 \leq \ell \leq n - j$ ), and zeros elsewhere. So  $GE_1 E_2 E_3 E_4 = I$ .

We follow a similar procedure to construct matrices for  $T(p)$ : For  $0 \leq r \leq n$ , we let  $\mathcal{G}_r$  be the set of matrices  $G$  so that for some ordered partition  $\mathcal{P} = (\{d_1, \dots, d_r\}, \{a_1, \dots, a_{n-r}\})$  of  $\{1, 2, \dots, n\}$ , for  $1 \leq t \leq r$ , column  $t$  of  $G$  has 1 in row  $d_t$  and zeros elsewhere, and for  $1 \leq t \leq n - r$ , the  $\ell, r + t$ -entry of  $G$  is  $\alpha_{\ell t}$  where  $\alpha_{\ell t}$  is 1 if  $\ell = a_t$ ,  $\alpha_{\ell t} = 0$  if  $\ell < a_t$  or  $\ell = a_i$  (some  $i \neq t$ ), and otherwise  $\alpha_{\ell t} \in \{0, 1, \dots, p-1\}$ . (So  $G \in GL_n(\mathbb{Z})$  with  $G^{-1} = E_1 E_2$  where  $E_1$  has  $i, i$ -entry 1 ( $1 \leq i \leq n$ ),  $\ell, r + t$ -entry  $-\alpha_{d_{\ell}t}$ , with zeros elsewhere, and  $E_2$  has  $t, d_t$ -entry 1 for  $1 \leq t \leq r$ ,  $r + t, a_t$ -entry 1 for  $1 \leq t \leq n - r$ , and zeros elsewhere.) Let  $\mathcal{Y}_r$  be the collection of matrices  $\begin{pmatrix} Y & \\ & 0 \end{pmatrix}$  where  $Y$  varies over integral  $r \times r$ , symmetric matrices modulo  $p$ , and let  $D_r = \begin{pmatrix} I_r & \\ & pI_{n-r} \end{pmatrix}$ .

**Theorem 2.** *Given a degree  $n$  Siegel modular form  $F$ ,*

$$F|T(p) = p^{n(k-n-1)/2} \sum_M F| \begin{pmatrix} \frac{1}{p}I_n & \\ d & I_n \end{pmatrix} M$$

where

$$M = \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^t G \end{pmatrix}$$

varies so that for some  $r$ ,  $0 \leq r \leq n$ ,  $D = D_r$ ,  $Y \in \mathcal{Y}_r$ , and  $G \in \mathcal{G}_r$ .

*Proof.* Using Proposition 3.1 of [3], we only need to show that as  $G$  varies over  $\mathcal{G}_r$ ,  $\Omega = \Lambda G p D_r^{-1}$  varies once over all lattices  $\Omega$  where  $p\Lambda \subseteq \Omega \subseteq \Lambda$ ,  $[\Lambda : \Omega] = p^r$ . So, similar to the proof of Theorem 1, we construct all the  $\Omega$  as well as a specific basis for each  $\Omega$ .

Let  $\bar{C}$  be a dimension  $n - r$  subspace of  $\Lambda/p\Lambda$ . Choose a basis  $\bar{v}_1, \dots, \bar{v}_{n-r}$  so that

$$\bar{v}_t = \bar{x}_{a_t} + \sum_{\ell > a_t} \bar{\alpha}_{\ell t} \bar{x}_{\ell}$$

where  $a_1, \dots, a_{n-r}$  are distinct,  $\bar{\alpha}_{\ell t} = 0$  if  $\ell = a_i$  (some  $i \neq t$ ); for each  $\bar{\alpha}_{\ell t}$ , take a preimage  $\alpha_{\ell t} \in \{0, 1, \dots, p-1\}$ . Then with  $(\{d_1, \dots, d_r\}, \{a_1, \dots, a_{n-r}\})$  an ordered partition of  $\{1, 2, \dots, n\}$  and  $G$  constructed according to our recipe preceding Theorem 2, we have  $\Omega = \Lambda G p D_r^{-1}$ .  $\square$

*Remark.* In [6] we discuss how a particular subgroup of  $Sp_n(\mathbb{Z})$  acts on Jacobi forms on  $f : \mathbb{H}_{(n-m)} \times \mathbb{C}^{n-m,m} \rightarrow \mathbb{C}$ . From this we see that for  $1 \leq j \leq n-m$ ,

$$f|T_j(p^2) = \sum_M f| \begin{pmatrix} \frac{1}{p} I_j & & & \\ & I_{n-j} & & \\ & & p I_j & \\ & & & I_{n-j} \end{pmatrix} M$$

where

$$M = \begin{pmatrix} D & Y \\ & D^{-1} \end{pmatrix} \begin{pmatrix} G^{-1} & \\ & {}^t G \end{pmatrix}$$

varies so that for some  $r_0, r_2$  with  $r_0 + r_2 \leq j$ ,  $D = D_{r_0, r_2}$ ,  $Y \in \mathcal{Y}'_{r_0, r_2}$ , and  $G \in \mathcal{G}_{\mathcal{P}}$  where  $\mathcal{P}$  is a partition of type  $(r_0, r_2)$  so that  $\{n-m+1, \dots, n\} \subseteq \{c_1, \dots, c_{n-j}\}$ . When  $n-m < j \leq n$ , the operator  $T_j(p^2)$  changes the index, as does  $T(p)$ , and the matrices giving the action of these operators is a bit more complicated; explicit matrices for these operators are given in [6].

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